

TOROIDAL BOARDS AND CODE COVERING

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ABSTRACT. A radius- r extend ball with center in an 1-dimensional vector subspace V of \mathbb{F}_q^3 is the set of elements of \mathbb{F}_q^3 with Hamming distance to V at most r . We define $c(q)$ as the size of a minimum covering of \mathbb{F}_q^3 by radius-1 extend balls. We define a quower as a piece of a toroidal chessboard that extends the covering range of a tower by the northeast diagonal containing it. Let $\xi_D(n)$ be the size of a minimal covering with quowers of an $n \times n$ toroidal board without a northeast diagonal. We prove that, for $q \geq 7$, $c(q) = \xi_D(q-1) + 2$. Moreover, our proof exhibits a method to build such covers of \mathbb{F}_q^3 from the quower coverings of the board. With this new method, we determine $c(q)$ for the odd values of q and improve both existing bounds for the even case.

Key words: Code covering, short covering, toroidal board, projective geometry.

1. INTRODUCTION

The problem of finding minimum coverings of \mathbb{F}_q^n with radius- r balls in the Hamming distance is classic in code theory. There is a book on the subject [3] and an updated table with the known bounds for the sizes of such coverings [7]. In [13], a variation of this problem was introduced: a radius- r **extend ball** with center in an 1-dimensional vector subspace V of \mathbb{F}_q^3 is the set of elements of \mathbb{F}_q^3 with Hamming distance to V at most r . We define $c_q(n, r)$ as the size of a minimum covering of \mathbb{F}_q^n by radius- r extend balls, such minimum covering is called a **short covering**.

In [10] some interesting reasons to study short coverings are listed. One is that short coverings were used to construct record breaking classical coverings. Other is that they respond well to some heuristic methods and give an economic way in terms of memory to store codes. A third one is that they seem to have more interesting mathematical properties than the classic coverings, like more compatibility with the algebraic structure of the vector space \mathbb{F}_q^n and connection with other structures (see [9, 11, 12] for examples).

Few values of $c_q(n, r)$ are known. Here, our concern are the values of $c(q) := c_q(3, 1)$. Some works [9, 12, 11] proved bounds for $c(q)$. In Corollary 4, we establish $c(q)$ for odd values of q and improve both existing bounds for the even values. In order to do this, we introduce a relation between short coverings, projective spaces and toroidal boards. The number of towers needed to cover an $n \times n$ toroidal board is well known, clearly n . Some studies on covering and packing of queens in the toroidal boards were made by [2] and [1]. We introduce a piece with range between a tower and a queen, as described next.

We will use $1, \dots, n$ as standard representatives for the classes of \mathbb{Z}_n . The toroidal $n \times n$ board will be modeled by \mathbb{Z}_n^2 , with the first coordinate indexing the column and the second the row, in such a way that $(1, 1)$ corresponds to the southwestern square and (n, n) to the northeastern square (the orientation is similar to a Cartesian plane). The **diagonal** of $(a, b) \in \mathbb{Z}_n^2$ is the set $D(a, b) := \{(a+t, b+t) : t \in \mathbb{Z}_n\}$. The **vertical** and **horizontal** lines of $(a, b) \in \mathbb{Z}_n^2$ are respectively defined by $V(a, b) := \{(a, t) : t \in \mathbb{Z}_n\}$ and $H(a, b) := \{(t, b) : t \in \mathbb{Z}_n\}$. The **quower** (an hybrid of queen and tower) of $(a, b) \in \mathbb{Z}_n^2$ is the set $QW(a, b) := D(a, b) \cup V(a, b) \cup H(a, b)$.

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We denote by \mathbb{D}_n the board without the northeast diagonal: $\mathbb{D}_n := \mathbb{Z}_n^2 - D(1, 1)$. We also denote by $\xi(n)$ and $\xi_D(n)$ the respective sizes of minimum coverings of \mathbb{Z}_n^2 and \mathbb{D}_n by quowers of \mathbb{Z}_n^2 . The next theorem establishes a relation between the values of $c(q)$ and $\xi_D(q-1)$:

Theorem 1. *For a prime power $q \geq 5$, $c(q) = \xi_D(q-1) + 2$. Moreover, for $q \geq 7$, there is an algorithm for building minimum coverings of \mathbb{F}_q^3 by radius-1 extended balls from coverings of \mathbb{D}_n by quowers and vice-versa.*

The proof of the first part of Theorem 1 is a construction that gives the algorithm for the second part. There are certain difficulties in using such technique for higher dimensions than 3. One is to make a more general version of Lemma 25 and other is to study the coverings of higher-dimensional boards. The next theorem establishes values and bounds for $\xi(n)$:

Theorem 2. *Let n be a positive integer.*

- (a) *If $n \equiv 2 \pmod{4}$, then $\xi(n) = n/2$.*
- (b) *If $n \equiv 0 \pmod{4}$, then $\xi(n) = 1 + n/2$.*
- (c) *If n is odd, then $\frac{n+1}{2} \leq \xi(n) \leq \frac{2n+1}{3}$.*

For values of $\xi_D(n)$, we have:

Theorem 3. *Let n be a positive integer.*

- (a) *If n is even, then $\xi_D(n) = n/2$.*
- (b) *If n is odd, then $\frac{n+1}{2} \leq \xi_D(n) \leq \frac{2n+1}{3}$.*

From Theorems 1 and 3 and the known values of $c(3)$ and $c(4)$ of [13] (see also Section 4), we have:

Corollary 4. *Let $q \geq 3$ be a prime power.*

- (a) *If q is odd, then $c(q) = \frac{q+3}{2}$.*
- (b) *If q is even, then $\frac{q+4}{2} \leq c(q) \leq \frac{2q+5}{3}$.*

The upper bound $c(q) \leq \frac{q+3}{2}$ in Corollary 4 was proved by Martinhão and Carmelo in [9] for $q \equiv 3 \pmod{4}$ and, in an unpublished work independent from this, for $q \equiv 1 \pmod{4}$. The upper bound in item (b) of Corollary 4 improves the previous one $c(q) \leq 6 \lceil \frac{q-1}{9} \rceil + 6 \lceil \log_4(\frac{q-1}{3}) \rceil + 3$, set in [11]. The lower bounds of Corollary 4 improves the bound $c(q) \geq (q+1)/2$ set in [12]. The next theorem give us better upper bounds for some even values of q :

Theorem 5. *For positive odd integers m and n :*

- (a) *$\xi(mn) \leq m\xi(n)$.*
- (b) *If $q := mn + 1 \geq 7$, then $c(q) \leq m\xi(n) + 2$.*

In Section 4, we use an ILP formulation to compute $\xi(n)$, $\xi_D(n)$ and $c(q)$ for small values of q and n not covered by our results. There are still few known values for $\xi(n)$ with n odd. Next, we state some conjectures:

Conjecture 6. *If p is a prime number, then $\xi(p) = \lfloor \frac{2p+1}{3} \rfloor$.*

Conjecture 7. *If n is an odd positive integer, then*

$$\xi(n) = \min\{(n/m)\xi(m) : m \text{ divides } n\}.$$

Conjecture 8. *If n is an odd positive integer, then*

$$\xi(n) = \min\{(n/p)\xi(p) : p \text{ is a prime divisor of } n\}.$$

Conjecture 9. *For n assuming positive integer values, $\lim_{n \rightarrow \infty} \frac{\xi(2n+1)}{2n+1} = \frac{2}{3}$.*

2. PROOFS OF THEOREMS 2, 3 AND 5

In this section, we prove Theorems 2, 3 and 5. We will prove some lemmas and establish some concepts first.

Next, we extend our definitions for more general groups than \mathbb{Z}_n . Let G be a finite group and $(a, b) \in G^2$. We define the respective **diagonal**, **vertical**, **horizontal** and **quower** of $(a, b) \in G^2$ as follows:

- $D(a, b) := D_G(a, b) := \{(ta, tb) : t \in G\}$;
- $H(a, b) := H_G(a, b) := \{(t, b) : t \in G\}$;
- $V(a, b) := V_G(a, b) := \{(a, t) : t \in G\}$;
- $QW(a, b) := QW_G(a, b) := D(a, b) \cup H(a, b) \cup V(a, b)$.

For $X \subseteq G^2$, we define $QW(X)$ as the union of all quowers in the form $QW(x)$ with $x \in X$. In an analogous way we define $D(X)$, $H(X)$ and $V(X)$.

We define $\mathbb{D}(G) := G^2 - D(1_G, 1_G)$ and denote by $\xi(G)$ and $\xi_D(G)$ the sizes of a minimum covering by quowers of G^2 and $\mathbb{D}(G)$ respectively. Suppose that $\varphi : G \rightarrow H$ is a group isomorphism and define $\Phi(a, b) = (\varphi(a), \varphi(b))$ for $(a, b) \in G$. It is clear that for $x \in G^2$, $QW_H(\Phi(x)) = \Phi(QW_G(x))$. Therefore:

Lemma 10. *If G and H are isomorphic groups, then $\xi(G) = \xi(H)$ and $\xi_D(G) = \xi_D(H)$.*

Lemma 11. *For each finite group G , $\xi_D(G) \geq (|G| - 1)/2$ and $\xi(G) \geq |G|/2$.*

Proof. Write $n := |G|$. Let $\{QW(x_1), \dots, QW(x_k)\}$ be a minimum covering of $\mathbb{D}(G)$ by quowers. It is clear that $k = \xi_D(G) \leq \xi(G) < n$. So, we may choose a vertical line L of G^2 avoiding $V(x_1), \dots, V(x_k)$. Note that $(L \cap \mathbb{D}(G)) - (H(x_1) \cup \dots \cup H(x_k))$ has at least $n - k - 1$ elements, which must be covered by $D(x_1), \dots, D(x_k)$. Since each diagonal intersects L in one element, then $k \geq n - k - 1$ and $\xi_D(G) = k \geq (n - 1)/2$. Analogously, we can prove that $\xi(G) \geq n/2$. \square

In the next Lemma, we define an auxiliary function δ . The proof is elementary.

Lemma 12. *Define a function $\delta : \mathbb{Z}_k^2 \rightarrow \mathbb{Z}_k$ by $\delta(a, b) = b - a$ for each $(a, b) \in \mathbb{Z}_k^2$. Then for $(a, b), (c, d) \in \mathbb{Z}_k^2$, $(c, d) \in D(a, b)$ if and only if $\delta(c, d) = \delta(a, b)$.*

Lemma 13. *If n is a positive integer and $n \equiv 2 \pmod{4}$, then $\xi_D(n) = \xi(n) = n/2$.*

Proof. By Lemma 11, it is enough to find a covering of \mathbb{Z}_n^2 with $n/2$ quowers. Define $X := \{(2, n), (4, n-2), \dots, (n, 2)\}$. Let us check that $\{QW(x) : x \in X\}$ covers the board. This covering is illustrated for $n = 6$ in Figure 1. Consider the function δ of Lemma 12 for $k = n$. Note that $\delta(X) = \{n-2, n-6, \dots, 2-n\}$. As $n \equiv 2 \pmod{4}$, it follows that $\delta(X)$ is the set of the even elements of \mathbb{Z}_n .

Now let $(a, b) \in \mathbb{Z}_n^2$. If both b and a are odd, then $\delta(a, b)$ is even and, therefore, $\delta(a, b) \in \delta(X)$ and $(a, b) \in D(X) \subseteq QW(X)$. Otherwise, if one of a or b is even, it is clear that (a, b) is the vertical or horizontal line of an element of X . Therefore, $\{QW(x) : x \in X\}$ covers the board and the lemma is valid. \square

Lemma 14. *For each positive integer n , $\xi_D(4n) = 2n$.*

Proof. By Lemma 11, it is enough to find a covering of \mathbb{D}_{4n} with $2n$ elements. Such covering is illustrated for $4n = 12$ in Figure 1. Define:

- $A = \{(2, 4n), (4, 4n-2), \dots, (2n, 2n+2)\}$,
- $B := \{(2n+2, 2n-2), (2n+4, 2n-4), \dots, (4n-2, 2)\}$ and
- $C := \{(4n, 2n)\}$

We claim that $\{QW(x) : x \in A \cup B \cup C\}$ covers \mathbb{D}_{4n} . Let $(a, b) \in \mathbb{D}_{4n}$. If a or b is even, then it is clear that (a, b) is in the horizontal or vertical line of a member of $A \cup B \cup C$. Suppose that both a and b are odd. We will use the function δ of Lemma 12 for $k = 4n$. So, $\delta(a, b)$ is

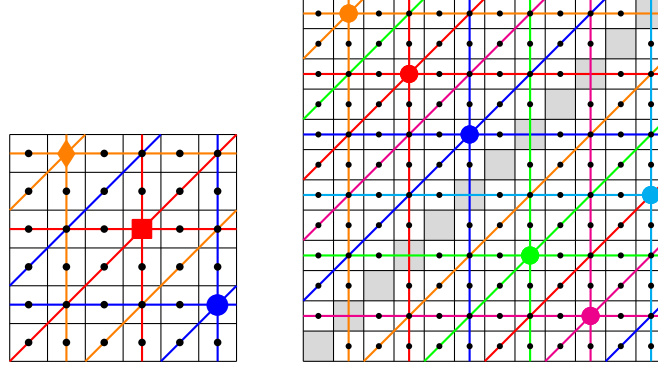


FIGURE 1. A covering of \mathbb{Z}_6^2 as in Lemma 13 and a covering of \mathbb{D}_{12} as in Lemma 14

even. Moreover, $\delta(a, b) \neq 0$, since $(a, b) \notin \mathbb{D}_n$ if $a = b$. Note that $\delta(A) = \{4n - 2, 4n - 6, \dots, 6, 2\}$ and $\delta(B) = \{-4, -8, \dots, 4 - 4n\}$. So, $\delta(A \cup B)$ contains all non-zero even elements of \mathbb{Z}_{4n} . In particular it contains $\delta(a, b) = b - a$. Therefore, $(a, b) \in D(x) \subseteq QW(x)$ for some $x \in A \cup B$ and the lemma holds. \square

The next Lemma is elementary and its proof is omitted.

Lemma 15. *If $\emptyset \subsetneq S \subsetneq \mathbb{Z}_n$ then $S \neq \{x + 1 : x \in S\}$.*

Lemma 16. *Let V_0 and V_1 be consecutive vertical lines of \mathbb{Z}_n^2 with $n \geq 2$. Suppose that $X, Y \subseteq \mathbb{Z}_n^2$ satisfy $|Y|, |X| \leq n - 1$ and $V_0 \cup V_1 \subseteq D(X) \cup H(Y)$. Then, $|X| + |Y| \geq n + 1$.*

Proof. Suppose the contrary. Say that $V_1 = \{(a + 1, b) : (a, b) \in V_0\}$. For $i = 0, 1$, V_i is the union of $A_i := V_i \cap D(X)$ and $B_i := V_i \cap H(Y)$. Note that $|A_0| = |A_1| = |X|$ and $|B_0| = |B_1| = |Y|$. For $i = 0, 1$, $|A_i| + |B_i| = |X| + |Y| \leq n = |V_i|$. So, $|A_i| \cap |B_i| = \emptyset$. Define a function $\pi : \mathbb{Z}_n^2 \rightarrow \mathbb{Z}_n$ by $\pi(a, b) = b$. As the restriction of π to each vertical line is bijective, then $\pi(A_0) = \mathbb{Z}_n - \pi(B_0) = \mathbb{Z}_n - \pi(B_1) = \pi(A_1)$. But $\pi(A_1) := \{t + 1 : t \in \pi(A_0)\}$. A contradiction to Lemma 15. \square

Lemma 17. *For each positive integer n , $\xi_D(2n + 1) \geq n + 1$.*

Proof. We may assume that $n \geq 1$. By Lemma 11, $\xi_D(2n + 1) \geq n$. Suppose for a contradiction that $\xi_D(2n + 1) = n$ and let $X := \{x_1, \dots, x_n\}$ be an n -subset of $\mathbb{D}(2n + 1)$ such that $\{QW(x) : x \in X\}$ covers $\mathbb{D}(2n + 1)$. Thus, there are two consecutive vertical lines V_0 and V_1 in \mathbb{Z}_{2n+1}^2 avoiding $V(X)$. As $\mathbb{D}(2n + 1) \subseteq QW(X)$, then $V_0 \cup V_1 \subseteq H(X) \cup D(X \cup \{(1, 1)\})$. By Lemma 16 $2n + 1 = 2|X| + 1 \geq (2n + 1) + 1$. A contradiction. \square

The next corollary follows from a theorem of Maillet [8], published in 1894 (a more accessible reference is Theorem 2 of [14]).

Corollary 18. *Let Q be a latin square with even order $n \geq 2$ symmetric in relation to one of its diagonals. Then, Q admits no set X of n entries such that each pair of entries of X are in different rows, different columns and has different symbols (such set is called a latin transversal).*

Lemma 19. *If n is a positive integer, then $\xi(4n) = 1 + 2n$.*

Proof. By Lemma 14, $2n = \xi_D(4n) \leq \xi(4n) \leq \xi_D(4n) + 1 = 2n + 1$. So, all we have to prove is that $\xi(4n) \neq 2n$. Suppose for a contradiction that $\xi(4n) = 2n$. Let $X := \{(a_t, b_t) : t = 1, \dots, 2n\}$ be a subset of \mathbb{Z}_{4n}^2 such that $\{QW(x) : x \in X\}$ covers \mathbb{Z}_{4n}^2 .

First we will prove that:

$$(1) \quad \{a_1, \dots, a_{2n}\}, \{b_1, \dots, b_{2n}\} \in \{\{1, 3, \dots, 4n - 1\}, \{2, 4, \dots, 4n\}\}.$$

Suppose the contrary. Then, there are two consecutive horizontal lines avoiding $H(X)$ or two consecutive vertical lines avoiding $V(X)$. We may assume the later case. Let V_0 and V_1

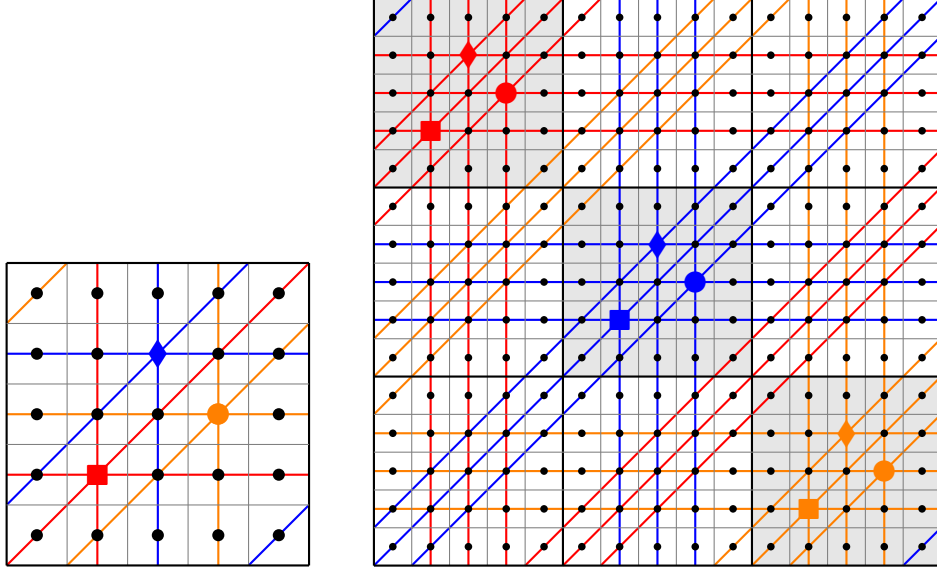


FIGURE 2. A covering of \mathbb{Z}_{15} by quowers constructed from a covering of \mathbb{Z}_5 with the method of the proof of Theorem 5

be such lines. So, $V_0 \cup V_1 \subseteq H(X) \cup D(X)$. By Lemma 16, $2|X| = 4n \geq 4n + 1$. A contradiction. So, (1) holds.

By (1), we may assume, without loss of generality, that $E := \{2, 4, \dots, 4n\} = \{a_1, \dots, a_{2n}\} = \{b_1, \dots, b_{2n}\}$. So $X \subseteq E \times E$. Let $F := \mathbb{Z}_n - E$. As $V(X) \cup H(X)$ does not intersect $F \times F$, hence $F \times F \subseteq D(X)$. We will use the function δ of Lemma 12 for $k = 4n$. Note that, in each row or column of $F \times F$, δ assumes $2n$ distinct values. By Lemma 12, δ also assumes $2n$ distinct values on X . Now, construct a latin square having E as set of rows and columns such that the symbol in (a, b) is $\delta(a, b)$. The entries of this latin square in X contradicts Lemma 18. \square

Next we prove Theorem 5.

Proof of Theorem 5: First, we prove item (a). An example with $m = 3$ and $n = 5$ is illustrated in Figure 2. We consider the board divided in m^2 blocks of size $n \times n$. We define a function $\varphi : \mathbb{Z}_{mn}^2 \rightarrow \mathbb{Z}_n^2$ in such a way that, for each p in a block B , $\varphi(p)$ is the element of \mathbb{Z}_n^2 in the corresponding position that p occupies in B . This is, $\varphi(a, b) = (a \bmod n, b \bmod n)$. Note that the restriction of φ to each block is a bijection $\varphi_B : B \rightarrow \mathbb{Z}_n^2$.

We call the blocks in the northwest diagonal the gray blocks (as in Figure 2). Let \mathcal{G} be the family of the gray blocks. Consider a minimum sized set $C \subseteq \mathbb{Z}_n^2$ such that $QW_{\mathbb{Z}_n}(C)$ covers \mathbb{Z}_n^2 . Define:

$$C^+ := \bigcup_{B \in \mathcal{G}} \varphi_B^{-1}(C).$$

For item (a), all we have to prove is that $QW_{\mathbb{Z}_{mn}}(C^+)$ covers \mathbb{Z}_{mn}^2 . Let $x \in \mathbb{Z}_{mn}^2$ and choose $y \in \mathbb{Z}_n^2$ such that $\varphi(x) \in QW_{\mathbb{Z}_n}(y)$. If $\varphi(x) \in V_{\mathbb{Z}_n}(y)$, then, for the gray block B meeting $V_{\mathbb{Z}_{mn}}(x)$, $x \in V_{\mathbb{Z}_{mn}}(\varphi_B^{-1}(y))$. If $\varphi(x) \in H_{\mathbb{Z}_n}(y)$, then for the gray block B meeting $H_{\mathbb{Z}_{mn}}(x)$, $x \in V_{\mathbb{Z}_{mn}}(\varphi_B^{-1}(y))$. Now, assume that $\varphi(x) \in D_{\mathbb{Z}_n}(y)$. We denote by δ_n the function of lemma 12 on \mathbb{Z}_n^2 and the same for δ_{mn}^2 and \mathbb{Z}_{mn} . Note that

$$\delta_{mn}(x) \equiv \delta_n(\varphi(x)) \bmod n \equiv \delta_n(y) \bmod n.$$

Define $Z = \{\varphi_B^{-1}(y) : B \in \mathcal{G}\}$. Hence, $\delta_{mn}(\varphi(z)) \equiv \delta_n(y) \bmod n$ for each $z \in Z$. Let $z_0 \in Z$. Then, $Z = \{z_0 + tn(-1, 1) : t \in \mathbb{Z}_{mn}\}$. This implies that $\delta_{mn}(Z) = \{\delta_{mn}(z_0) + 2tn : t \in \mathbb{Z}_{mn}\}$.

Since mn is odd, then $\delta_{mn}(Z) = \{\alpha \in \mathbb{Z}_{mn} : \alpha \equiv \delta_{mn}(z_0) \pmod{n}\}$. As

$$\delta_{mn}(x) \equiv \delta_n(y) \pmod{n} \equiv \delta_{mn}(z_0) \pmod{n},$$

then $\delta_{mn}(x) \in \delta_{mn}(Z)$ and $x \in D_{\mathbb{Z}_{mn}}(Z)$. As $Z \subseteq C^+$, $x \in QW_{\mathbb{Z}_{mn}}(C^+)$ and item (a) holds. Item (b) follows from item (a) and Theorem 1. \square

Lemma 20. *If n is odd, then $\xi(n) \leq \lfloor (2n+1)/3 \rfloor$.*

Proof. Write $n = 3m + r$ with $r \in \{0, 1, 2\}$. If $r = 0$, then, as $\xi(3) = 2$, by Theorem 5, $\xi(n) \leq 2m = \lfloor (2n+1)/3 \rfloor$ and the lemma holds. So, assume that $r \in \{1, 2\}$. We shall prove that $\xi(n) \leq 2m+1$. Consider a subdivision of the board as below:

$Q_{1,3}$	$Q_{2,3}$	$Q_{3,3}$
$Q_{1,2}$	$Q_{2,2}$	$Q_{3,2}$
$Q_{1,1}$	$Q_{2,1}$	$Q_{3,1}$

where $Q_{1,1}$, $Q_{2,2}$ and $Q_{3,3}$ are square blocks with respective orders $m+1$, m and $m+r-1$. Let X_i be the set of the pairs of \mathbb{Z}_n^2 in the northwest diagonal of $Q_{i,i}$. Define $X = X_1 \cup X_2$. We will prove that $\{QW(x) : x \in X\}$ covers the board. If $Q_{i,j} \neq Q_{3,3}$ it is clear that $Q_{i,j} \subseteq \bigcup_{x \in X} (H(x) \cup V(x))$. So, let $y \in Q_{3,3}$. We shall prove that $y \in D(x)$ for some $x \in X$. We will use the function δ of Lemma 12 for $k = n$. If c is a coordinate of y , then $2m+2 \leq c \leq 3m+r$. Therefore, $-m \leq -m-r+2 \leq \delta(y) \leq m+r-2 \leq m$. Note that $\delta(X_1) = \{m, m-2, m-4, \dots, 4-m, 2-m, -m\}$ and $\delta(X_2) = \{m-1, m-3, \dots, 3-m, 1-m\}$. So, $\delta(X_1 \cup X_2) = \{m, m-1, \dots, 1-m, -m\}$ and $\delta(y) \in \delta(X)$. By Lemma 12, y is in $D(x)$ for some $x \in X$ and the lemma holds. \square

Proof of Theorem 2: Items (a) and (b) follows from Lemmas 13 and 19, respectively. Item (c) follows from Lemmas 17 and 20. \square

Proof of Theorem 3: Items (a) and follows from Lemmas 13 and 14. Item (c) follows from Lemmas 17 and 20. \square

3. FROM \mathbb{F}_q^3 TO THE PROJECTIVE PLANE

In this section, we establish relations between short coverings and coverings by quowers using the projective plane as a link between them. In the end of the section, Theorem 1 is proved.

We define the projective plane $PG(2, q)$ as the geometry whose points are the 1-dimensional vector subspaces of \mathbb{F}_q^3 and lines are the sets of points whose union of members are 2-dimensional subspaces of \mathbb{F}_q^3 . We denote the subspace spanned by $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3 - \{0\}$ by homogeneous coordinates $(\alpha : \beta : \gamma) \in PG(2, q)$.

We say that the points of $PG(2, q)$ are **cardinal**, **coast** or **midland** when they have exactly one, two or three non-zero coordinates respectively. We denote the cardinal points by $c_1 := (1 : 0 : 0)$, $c_2 := (0 : 1 : 0)$ and $c_3 := (0 : 0 : 1)$. We also denote the line containing the points u and v by $\overleftrightarrow{u, v}$, provided $u \neq v$, and, for convenience, $\overleftrightarrow{u, u} := \{u\}$. We say that a line of $PG(2, q)$ is a **midland line** if it contains a midland point and a **coast line** otherwise. Moreover, we denote by e_i the i -th vector in the canonical basis of \mathbb{F}_q^3 , and by $[v_1, \dots, v_n]$ the subspace of \mathbb{F}_q^3 spanned by v_1, \dots, v_n . The next lemma is easy to check:

Lemma 21. *Let $v \in \mathbb{F}_q^3 - \{0\}$. Then, $B_E[v, 1]$ is the union of the members of $\overleftrightarrow{[v], c_1} \cup \overleftrightarrow{[v], c_2} \cup \overleftrightarrow{[v], c_3}$.*

Motivated by Lemma 21, we define the **wind rose** of $p \in PG(2, q)$ as

$$W(p) := \overleftrightarrow{p, c_1} \cup \overleftrightarrow{p, c_2} \cup \overleftrightarrow{p, c_3}.$$

From Lemma 21, we may conclude:

Corollary 22. *Let $p_1, \dots, p_n \in PG(2, q)$ and for $i = 1, \dots, n$, let $v_i \in p_i - \{0\}$. Then, $\{W(p_1), \dots, W(p_n)\}$ covers $PG(2, q)$ if and only if $\{B_E[v_1, 1], \dots, B_E[v_n, 1]\}$ covers \mathbb{F}_q^3 .*

Corollary 22 implies that $c(q)$ is the size of a minimum covering of $PG(2, q)$ by wind roses. We say that a wind rose $W(p)$ is **cardinal**, **coast** or **midland** according to what of these adjectives applies to p . It is clear that $W(p_1) = W(p_2)$ implies $p_1 = p_2$. So, each wind rose fits in only one of such adjectives. The following properties of wind roses are elementary and easy to check:

Lemma 23. *Each midland wind rose is the union of three distinct midland lines, each coast wind rose is the union of a coast and a midland line and each cardinal wind rose is the union of two distinct coast lines.*

Lemma 24. *Let q be a prime power and suppose that $c(q) \leq q - 2$. Then, every minimum covering \mathcal{C} of $PG(2, q)$ by wind roses contains at least two non midland wind roses. In particular, each coast line is contained in a member of \mathcal{C} .*

Proof. Since $PG(2, q)$ has three distinct coast lines, the first part of the lemma follows from the second part and from Lemma 23. So, let us prove the second part. Suppose that it fails. Let \mathcal{C} be a minimum covering of $PG(2, q)$ by wind roses such that no member contains a fixed coast line L . Let K be the set of coast points in L . Since no member of \mathcal{C} contains L , then each wind rose in \mathcal{C} meets K in at most one point. So, $q - 1 = |K| \leq |\mathcal{C}| = c(q) \leq q - 2$. A contradiction. \square

Lemma 25. *Let q be a prime power and suppose that $c(q) \leq q - 2$. Then, there is a minimum covering of $PG(2, q)$ by wind roses containing precisely one cardinal wind rose and one coast wind rose.*

Proof. Chose a minimum covering \mathcal{C} of $PG(2, q)$ by wind roses maximizing the number of midland wind roses primarily and coast wind roses secondarily. There are three coast lines in $PG(2, q)$: the members of $\mathcal{L} := \{\overrightarrow{c_1, c_2}, \overrightarrow{c_1, c_3}, \overrightarrow{c_2, c_3}\}$. By Lemmas 24 and 23, the members of \mathcal{L} are covered by:

- (i) One cardinal and one coast wind rose of \mathcal{C} ,
- (ii) Two cardinal wind roses of \mathcal{C} , or
- (iii) Three coast wind roses of \mathcal{C} .

We shall prove that (i) occurs. Indeed, first suppose for a contradiction that (ii) holds. Say that the members of \mathcal{L} are covered by $W(c_1)$ and $W(c_2)$. If p is a coast point of $\overrightarrow{c_2, c_3}$, then $W(c_1)$ and $W(p)$ are enough to cover the coast lines of $PG(2, q)$. Hence $(\mathcal{C} - W(c_2)) \cup W(p)$ contradicts the secondary maximality of \mathcal{C} . Thus (ii) does not hold.

Now, suppose that (iii) holds. The coast lines of $PG(2, q)$ are covered by three coast wind roses $W(p_1), W(p_2), W(p_3) \in \mathcal{C}$. It is clear that p_1, p_2 and p_3 are in different coast lines. For $\{i, j, k\} = \{1, 2, 3\}$, say that $p_k \in \overrightarrow{c_i, c_j}$. Let x be the intersection point of $\overrightarrow{c_2, p_2}$ and $\overrightarrow{c_3, p_3}$. Note that x is a midland point. We claim that

$$\mathcal{C}' := (\mathcal{C} - \{W(p_2), W(p_3)\}) \cup \{W(c_1), W(x)\}$$

contradicts the primary maximality of \mathcal{C} . Note that \mathcal{C}' has more midland wind roses than \mathcal{C} and $|\mathcal{C}'| \leq |\mathcal{C}|$. It is left to show that \mathcal{C}' covers $PG(2, q)$. For this purpose, it is enough to prove that $W(p_2) \cup W(p_3) \subseteq W(c_1) \cup W(x)$. Indeed, since $p_2 \in \overrightarrow{c_1, c_3}$, then $W(p_2) = \overrightarrow{c_1, c_3} \cup \overrightarrow{c_2, p_2}$, but $\overrightarrow{c_2, p_2} = \overrightarrow{c_2, x} \subseteq W(x)$ and $\overrightarrow{c_1, c_3} \subseteq W(c_1)$. Moreover, $W(p_3) = \overrightarrow{c_1, c_2} \cup \overrightarrow{c_3, p_3}$, but $\overrightarrow{c_3, p_3} = \overrightarrow{c_3, x} \subseteq W(x)$ and $\overrightarrow{c_1, c_2} \subseteq W(c_1)$. So, \mathcal{C}' covers $PG(2, q)$ and (iii) does not occur. Therefore, (i) holds.

Now, let W_1 and W_2 be the respective wind roses described in (i). It is left to prove that W is midland if $W \in \mathcal{C} - \{W_1, W_2\}$. Since all cardinal wind roses are contained in $W_1 \cup W_2$, then,

by the minimality of \mathcal{C} , W is not cardinal. If W is coast, then W is the union of a coast line C and a midland line M . But, $C \subseteq W_1 \cup W_2$ and, for a midland point $x \in M - C$, $M \subseteq W(x)$. Then $(\mathcal{C} - W) \cup W(x)$ violates the primary maximality of \mathcal{C} . Therefore, W is midland and the lemma holds. \square

We define a bijection $f : PG(2, q) \rightarrow PG(2, q)$ to be a **projective isomorphism** if $f(\overleftrightarrow{xy}) = \overleftrightarrow{f(x)f(y)}$ for all $x, y \in PG(2, q)$.

Lemma 26. *If f is a projective automorphism of $PG(2, q)$ carrying cardinal points into cardinal points, then $f(W(x)) = W(f(x))$ for each $x \in PG(2, q)$. Moreover, x is midland (resp. coast, cardinal) if and only if so is $f(x)$.*

Proof. For $x \in PG(2, q)$:

$$\begin{aligned} f(W(x)) &= f(\overleftrightarrow{xc_1} \cup \overleftrightarrow{xc_2} \cup \overleftrightarrow{xc_3}) \\ &= f(\overleftrightarrow{xc_1}) \cup f(\overleftrightarrow{xc_2}) \cup f(\overleftrightarrow{xc_3}) \\ &= \overleftrightarrow{f(x)f(c_1)} \cup \overleftrightarrow{f(x)f(c_2)} \cup \overleftrightarrow{f(x)f(c_3)} \\ &= \overleftrightarrow{f(x)c_1} \cup \overleftrightarrow{f(x)c_2} \cup \overleftrightarrow{f(x)c_3} \\ &= W(f(x)). \end{aligned}$$

This proves the first part of the lemma. For the second part, by hypothesis, x is cardinal if and only if $f(x)$ is cardinal. Also, x is coast if and only if x is not cardinal but is in the line containing two cardinal points and so is $f(x)$. Therefore, x is coast if and only if so is $f(x)$. By elimination, this implies that x is midland if and only if so is $f(x)$. \square

The next Lemma has an straightforward proof.

Lemma 27. *Let x be a cardinal point and y a coast point of $PG(2, q)$ such that $W(x) \cup W(y)$ contains all coast points of $PG(2, q)$. Then, for some $\{i, j, k\} = \{1, 2, 3\}$, $x = c_i$ and $y \in \overleftrightarrow{c_j c_k}$.*

Lemma 28. *Let q be a prime power and suppose that $c(q) \leq q - 2$. There is a minimum covering of $PG(2, q)$ by wind roses containing $W(0 : 0 : 1)$ and $W(1 : 1 : 0)$ and such that all other members are midland.*

Proof. By Lemma 25, there is a minimum covering \mathcal{C} of $PG(2, q)$ by wind roses, whose all wind roses are midland, except for two, namely $W(x)$ and $W(y)$, where x is a cardinal point and y a coast point. We may define a projective automorphism $f : PG(2, q) \rightarrow PG(2, q)$ by permutations of homogeneous coordinates and multiplying a fixed coordinates by non-zero factors such that $f(x) = (0 : 0 : 1)$. By Lemma 27, $f(y)$ is in the form $(a : b : 0)$ with $a \neq 0 \neq b$. So, in addition, we may pick f in such a way that $f(y) = (1 : 1 : 0)$. By Lemma 26, $\{f(W) : W \in \mathcal{C}\}$ is the covering we are looking for. \square

Consider the multiplicative group \mathbb{F}_q^* , the set M of midland points in $PG(2, q)$ and the bijection ψ between $(\mathbb{F}_q^*)^2$ and M defined by $\psi(a, b) = (a : b : 1)$. For $x = (a, b) \in (\mathbb{F}_q^*)^2$, we clearly have $\psi(H(x)) = M \cap \overleftrightarrow{\psi(x)c_1}$ and $\psi(V(x)) = M \cap \overleftrightarrow{\psi(x)c_2}$. Moreover, as $D(x) = \{(ta, tb) : t \in \mathbb{F}_q^*\}$, hence:

$$\psi(D(x)) = \{(ta : tb : 1) : t \in \mathbb{F}_q^*\} = \{(a : b : t^{-1}) : t \in \mathbb{F}_q^*\} = M \cap \overleftrightarrow{\psi(x)c_3}.$$

As a consequence, $\psi(QW(x)) = M \cap W(\psi(x))$. Note that $\psi(D(1, 1)) = M \cap \overleftrightarrow{c_3, (1 : 1 : 0)}$. Therefore, the following lemma holds:

Lemma 29. *Consider the function ψ as defined above and let $X \subseteq (\mathbb{F}_q^*)^2$. Hence, $QW(X)$ is a covering by quowers of $\mathbb{D}(\mathbb{F}_q^*)$ if and only if $\{W(\psi(x)) : x \in X\} \cup \{(0 : 0 : 1), (1 : 1 : 0)\}$ is a covering of $PG(2, q)$ by wind roses.*

Now, we are in conditions to prove Theorem 1.

Proof of Theorem 1: It is well known that the multiplicative group of a finite field is cyclic. Thus, by Lemma 10, $\xi_D(\mathbb{F}_q^*) = \xi_D(q-1)$. By Lemmas 28 and 29, the result is valid when $c(q) \leq q-2$. When $q=5$ the values are known and matches the theorem (see section 4). So, assume that $q \geq 7$. We just have to prove that $c(q) \leq q-2$. Let \mathcal{Q} be a minimum covering of $(\mathbb{F}_q^*)^2$ by quowers and \mathcal{W} the covering of $PG(2, q)$ by wind roses obtained from \mathcal{Q} as in Lemma 29. Define $n := q-1$. So, $|\mathcal{W}| - 2 = |\mathcal{Q}| = \xi_D(\mathbb{F}_q^*) = \xi_D(n)$. Let us check that $\xi_D(n) \leq n-3$ for $n \geq 6$. For even values of n and for odd values greater than 7, it follows from Theorem 3. But it is known that $\xi_D(7) = 4$ (see section 4). So, $\xi_D(n) \leq n-3$ for $n \geq 6$. Therefore, $c(q) \leq |\mathcal{W}| \leq \xi_D(n) + 2 \leq n-1 = q-2$ and the theorem holds. \square

4. PARTICULAR INSTANCES AND ILP FORMULATION

For $X \in \{\mathbb{Z}_n^2, \mathbb{D}_n\}$, the following integer 0-1 linear program may be used to find minimum coverings of X by quowers of \mathbb{Z}_n^2 . In this formulation, $x_p = 1$ if and only if $QW(p)$ is used in the covering.

$$\begin{aligned} \text{Minimize : } & \sum_{p \in \mathbb{Z}_n^2} x_p \\ \text{Subject to : } & \forall q \in X : \sum_{p \in \mathbb{Z}_n^2 : q \in QW(p)} x_p \geq 1. \end{aligned}$$

For finding short coverings of \mathbb{F}_q^3 , a formulation in terms of wind roses in $PG(2, q)$ works similarly (see Corollary 22):

$$\begin{aligned} \text{Minimize : } & \sum_{p \in PG(2, q)} x_p \\ \text{Subject to : } & \forall q \in PG(2, q) : \sum_{p \in PG(2, q) : q \in W(p)} x_p \geq 1. \end{aligned}$$

Some instances not covered by our theorems were solved using GLPK [4], Cplex [6] and Gurobi [5]. They are displayed in the tables below. The values $c(2)$, $c(3)$ and $c(4)$ are already known from [12].

n	3	5	7	9	11	13	15	17
$\xi(n)$	2	3	5	6	7	9	9	11
$\xi_D(n)$	2	3	4	6	7	8	9	11

q	2	3	4	5	8	16
$c(q)$	1	3	3	4	6	11

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